

On Euler polynomials for projective hypersurfaces

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Abstract

For every positive integer $n \in \mathbb{Z}_+$ we define an ‘Euler polynomial’ $\mathcal{E}_n(t) \in \mathbb{Z}[t]$, and observe that for a fixed n all Chern numbers (as well as other numerical invariants) of all smooth hypersurfaces in \mathbb{P}^n may be recovered from the single polynomial $\mathcal{E}_n(t)$. More generally, we show that all Chern classes of hypersurfaces in a smooth variety may be recovered from its top Chern class.

Fix an algebraically closed field \mathfrak{K} of characteristic zero. We denote by \mathbb{P}^n projective n -space over \mathfrak{K} and for a smooth subvariety X of \mathbb{P}^n we define its Euler characteristic denoted $\chi(X)$ to be $\int_X c(TX) \cap [X]$ ¹. For every positive integer $n \in \mathbb{Z}_+$, there exists a polynomial $\mathcal{E}_n(t) \in \mathbb{Z}[t]$ such that if $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree d , then

$$\chi(X) = \mathcal{E}_n(d).$$

The adjunction formula along with standard exact sequences yield

$$\mathcal{E}_n(t) = (-1) \sum_{k=0}^{n-1} \binom{n+1}{k} (-t)^{n-k},$$

which we refer to as the n th *Euler polynomial*. In this note we make the observation that not just the Euler characteristic, but *all* Chern numbers of *all* smooth projective hypersurfaces in \mathbb{P}^n (along with all Euler characteristics of all their general hyperplane sections) may be recovered from $\mathcal{E}_n(t)$ (for all n).

Remark 0.1. More generally, for $X \subset \mathbb{P}^n$ a (possibly) *singular* hypersurface of degree d over \mathbb{C} , it is known that

$$(0.1) \quad \chi(X) = \mathcal{E}_n(d) + \int_X \mathcal{M}(X),$$

¹For $\mathfrak{K} = \mathbb{C}$ certainly $\chi(X) = \chi_{\text{top}}(X)$.

where $\chi(X)$ here denotes topological Euler characteristic with compact supports, and $\mathcal{M}(X)$ denotes the *Milnor class*² of X , a characteristic class supported on the singular locus of X (so that $\mathcal{M}(X) = 0$ for X smooth). As X is in the same rational equivalence class as a smooth hypersurface of degree d , the Milnor class then measures the deviation of $\chi(X)$ from that of a smooth deformation (parametrized by \mathbb{P}^1). As the Milnor class is defined for any \mathfrak{K} -variety, in this more general setting we *define* the Euler characteristic of a possibly singular hypersurface X to be $\chi(X) := \int_X (T_{\text{vir}}(X) + \mathcal{M}(X))$, where $T_{\text{vir}}(X)$ denotes the virtual tangent bundle of X . Thus formula (0.1) holds without the assumption $\mathfrak{K} = \mathbb{C}$.

So let R be a ring and let $\vartheta : R[t] \rightarrow R[t]$ be the map which keeps only the terms of degree greater than one, and then divides the result by $-t$, i.e., the map given by

$$a_n t^n + \cdots + a_0 \mapsto -(a_n t^{n-1} + \cdots + a_2 t).$$

Our result is the following

Theorem 0.1. *Let $n \in \mathbb{Z}_+$ and*

$$\mathfrak{C}_n(s, t) = \vartheta^{n-1} \mathcal{E}_n(t) s + \vartheta^{n-2} \mathcal{E}_n(t) s^2 + \cdots + \vartheta \mathcal{E}_n(t) s^{n-1} + \mathcal{E}_n(t) s^n \in \mathbb{Z}[s, t],$$

where ϑ^k denotes the k -fold composition of the map $\vartheta : \mathbb{Z}[t] \rightarrow \mathbb{Z}[t]$. Then

$$\mathfrak{C}_n(H, d) = \iota_* c(TX) \cap [X],$$

where $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree d , $H = c_1(\mathcal{O}_{\mathbb{P}^n}(1)) \cap [\mathbb{P}^n]$ and $\iota : X \hookrightarrow \mathbb{P}^n$ is the natural inclusion.

We then immediately arrive at the following

Corollary 0.2. *Let X be a hypersurface of degree d in \mathbb{P}^n . Then all Chern numbers of X are of the form*

$$\prod_i \mathcal{E}_n^{j_i}(d),$$

where $\sum j_i = n - 1$ and $\mathcal{E}_n^k(t)$ denotes $\vartheta^{n-(k+1)} \mathcal{E}_n(t)$.

Theorem 0.1 actually follows from a more general fact about hypersurfaces, which is nothing more than an elementary observation about the *Fulton class* of a hypersurface. Before stating the result we need the following

²For more on Milnor classes from an algebraic perspective see [3].

Definition 0.3. Let M be a smooth \mathfrak{K} -variety and $Y \hookrightarrow M$ be a regular embedding. The *Segre class* of Y relative to M is then given by

$$s(Y, M) := c(N_Y M)^{-1} \cap [Y] \in A_* Y,$$

where $N_Y M$ denotes the normal bundle to Y in M . The *Fulton class* of Y is then given by

$$c_F(Y) := c(TM) \cap s(Y, M).$$

Remark 0.2. In [5] (Chapter 4), Fulton actually defines the relative Segre class for Y an arbitrary *subscheme* of M , and proves the class $c(TM) \cap s(Y, M)$ is intrinsic to Y (i.e., it is independent of its embedding into a smooth variety). However, as we consider only hypersurfaces in this note (which are *always* regularly embedded), we don't need the definition in its full generality. In any case, for Y smooth $c_F(Y)$ coincides with its usual Chern class.

The previous results stated in this note are consequences of the following

Theorem 0.4. Let M be a smooth \mathfrak{K} -variety of dimension n , denote its Chern classes by c_i and let $\mathcal{E}_n(s) \in A_* M[s]$ be given by

$$\mathcal{E}_n(s) = c_{n-1}s - c_{n-2}s^2 + \cdots + (-1)^n c_1 s^{n-1} + (-1)^{n+1} s^n.$$

Then if X is any hypersurface in M we have

$$(0.2) \quad c_F(X) = \vartheta^{n-1} \mathcal{E}_n(X) + \vartheta^{n-2} \mathcal{E}_n(X) + \cdots + \vartheta \mathcal{E}_n(X) + \mathcal{E}_n(X),$$

where $c_F(X)$ denotes the ‘Fulton class’ of X , and X on the RHS of formula (0.2) denotes its class in $A_* M$. In particular, all Fulton classes of X may be recovered from its top Fulton class via the map ϑ .

Proof. Let X be a (possibly singular) hypersurface in a smooth variety M and denote its class in $A_* M$ simply by X . Then its Fulton class is defined as $c(TM) \cap s(X, M)$, where $s(X, M)$ denotes the Segre class of X in M . Since X is a hypersurface, it is regularly embedded, thus $s(X, M) = \frac{X}{1+X}$. Its Fulton classes (i.e., Chern classes for X smooth) then take the following form (as a class in the Chow group $A_* M$):

$$\begin{aligned} c_0(X) &= X \\ c_1(X) &= c_1 X - X^2 \\ c_2(X) &= c_2 X - c_1 X^2 + X^3 \\ &\vdots \\ c_{n-1}(X) &= c_{n-1} X - c_{n-2} X^2 + \cdots + (-1)^n c_1 X^{n-1} + (-1)^{n+1} X^n = \mathcal{E}_n(X), \end{aligned}$$

where by X^k we mean the k -fold intersection product of X with itself. The theorem immediately follows. \square

Theorem 0.1 may then be obtained by replacing M by \mathbb{P}^n and X by $tc_1(\mathcal{O}_{\mathbb{P}^n}(1)) \cap [\mathbb{P}^n]$.

Remark 0.3. Not only the Chern numbers, but the Euler characteristics of all general hyperplane sections of all hypersurfaces in \mathbb{P}^n may be easily recovered from $\mathcal{E}_n(t)$ as well. More precisely, let

$$\mathfrak{C}_n^\vee(s, t) := s^n \mathfrak{C}_n\left(\frac{1}{s}, t\right) = \mathcal{E}_n(t) + \vartheta \mathcal{E}_n(t)s + \cdots + \vartheta^{n-1} \mathcal{E}_n(t)s^{n-1},$$

where $\mathfrak{C}_n(s, t)$ is as given in the statement of Theorem 0.1 (i.e., the power of s is now keeping track of dimension rather than codimension), and let

$$\mathfrak{e}_n(s, t) := \frac{s \cdot \mathfrak{C}_n^\vee(-s-1, t) + \mathfrak{C}_n^\vee(0, t)}{s+1}.$$

Then by Theorem 1.1 in [1], if $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree d then the coefficient of $(-s)^r$ in $\mathfrak{e}_n(s, d)$ is the Euler characteristic of $X \cap H_1 \cap \cdots \cap H_r$, where the H_i are general hyperplanes with respect to $X \cap H_1 \cap \cdots \cap H_{i-1}$.

Remark 0.4. For X a smooth \mathfrak{K} -variety it was shown in [4] that the (unnormalized) *motivic Hirzebruch class* of X^3 (referred to as the “Hirzebruch series” in [loc. cit.]), denoted $T_y^*(X)$, may be given by

$$(0.3) \quad T_y^*(X) = \frac{(1+y)^k}{k!} \frac{d^k}{ds^k} \exp\left(\ln\left(\frac{s(1+ye^{-s})}{1-e^{-s}}\right) \odot \frac{-sC'}{C}\right) \Big|_{s=0},$$

where $k = \dim(X)$, $C = 1 - c_1(X)s + \cdots + (-1)^k c_k(X)s^k$, $C' = \frac{d}{ds}C$, and \odot denotes the *Hadamard product* of power series⁴. Thus for $X \subset \mathbb{P}^n$ a smooth hypersurface of degree d , replacing C by $\mathfrak{C}_n(-sH, d)$ in formula (0.3) yields a formula for the motivic Hirzebruch class of X in terms of $\mathcal{E}_n(t)$, where \mathfrak{C}_n and H are as given in Theorem 0.1. The degree zero part of $T_y^*(X)$ then yields the Euler characteristic, arithmetic genus and signature of X when evaluated at $y = -1, 0, 1$, respectively. For $\mathfrak{K} = \mathbb{C}$ and X hyperkähler, a geometric interpretation for arbitrary y is given in [6].

Example 0.5. Let $n = 4$. Then

$$\begin{aligned} \mathcal{E}_4(t) &= 10t - 10t^2 + 5t^3 - t^4 \\ \vartheta \mathcal{E}_4(t) &= 10t - 5t^2 - t^3 \\ \vartheta^2 \mathcal{E}_4(t) &= 5t - t^2. \end{aligned}$$

³Motivic Hirzebruch classes were first defined in [2]. For a pedagogic introduction to motivic characteristic classes we recommend [7].

⁴For $f = \sum a_i s^i$ and $g = \sum b_i s^i$, then $f \odot g = \sum a_i b_i s^i$.

Thus for $X \subset \mathbb{P}^4$ a smooth hypersurface of degree d by Corollary 0.2 we have

$$\begin{aligned} c_1(X)^3 &= (\vartheta^2 \mathcal{E}_4(d))^3 = (5d - d^2)^3 \\ c_1(X)c_2(X) &= \vartheta^2 \mathcal{E}_4(d) \cdot \vartheta \mathcal{E}_4(d) = (5d - d^2)(10d - 5d^2 - d^3) \\ c_3(X) &= \mathcal{E}_4(d) = 10d - 10d^2 + 5d^3 - d^4. \end{aligned}$$

Moreover, the Lefschetz hyperplane theorem and Hirzebruch-Riemann-Roch yields

$$\begin{aligned} h^{0,3}(X) &= 1 - \frac{\vartheta^2 \mathcal{E}_4(d) \cdot \vartheta \mathcal{E}_4(d)}{24}, \\ h^{1,2}(X) &= \frac{\vartheta^2 \mathcal{E}_4(d) \cdot \vartheta \mathcal{E}_4(d)}{24} - \frac{\mathcal{E}_4(d) - 2}{2}, \end{aligned}$$

which are the only nontrivial Hodge numbers of X . Furthermore, we have

$$\begin{aligned} \mathfrak{e}_4(s, d) &= \mathcal{E}_4(d) + (-\vartheta \mathcal{E}_4(d) + \vartheta^2 \mathcal{E}_4(d) - \vartheta^3 \mathcal{E}_4(d))s \\ &\quad + (\vartheta^2 \mathcal{E}_4(d) - 2\vartheta^3 \mathcal{E}_4(d))s^2 + \vartheta^4 \mathcal{E}_4(d)s^3 \\ &= (10d - 10d^2 + 5d^3 - d^4) + (-6d + 4d^2 - d^3)s + (3d - d^2)s^2 - ds^3, \end{aligned}$$

illustrating the fact that all Chern numbers, all Hodge numbers and all Euler characteristics of general hyperplane sections of all smooth hypersurfaces in \mathbb{P}^4 may be obtained via $\mathcal{E}_4(t)$ and the map ϑ .

We end with the following

Question 1. *Is there a more general class of varieties (besides hypersurfaces) for which all of its Chern classes may be recovered from its top Chern class?*

Acknowledgements. We thank Paolo Aluffi for pointing us in the more general direction of Theorem 0.4, from which the proof of Theorem 0.1 immediately follows.

REFERENCES

- [1] P. Aluffi. Euler characteristics of general linear sections and polynomial Chern classes. *Rend. Circ. Mat. Palermo (2)*, 62(1):3–26, 2013.
- [2] Jean-Paul Brasselet, Jörg Schürmann, and Shoji Yokura. Hirzebruch classes and motivic Chern classes for singular spaces. *J. Topol. Anal.*, 2(1):1–55, 2010.
- [3] J. Fullwood. On Milnor classes via invariants of singular schemes. *Journal of singularities*, 8:1–10, 2014.
- [4] J. Fullwood and M. van Hoeij. On Hirzebruch invariants of elliptic fibrations. In *String-Math 2011*, volume 85 of *Proc. Sympos. Pure Math.*, pages 355–366. Amer. Math. Soc., Providence, RI, 2012.
- [5] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*. Springer-Verlag, Berlin, second edition, 1998.
- [6] G. Thompson. A geometric interpretation of the χ_y genus on hyper-Kähler manifolds. *Comm. Math. Phys.*, 212(3):649–652, 2000.
- [7] Shoji Yokura. Motivic characteristic classes. In *Topology of stratified spaces*, volume 58 of *Math. Sci. Res. Inst. Publ.*, pages 375–418. Cambridge Univ. Press, Cambridge, 2011.